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Fixed-Point Theorems Involving P-Compact, Semicontractive, and Accretive Operators Not Defined on All of a Banach Space

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INTRODUCTION

Let X be a real Banach space, D a closed ball or a closed bounded convex subset of X with $0 \in \mathrm{Int}(D)$ and A a nonlinear mapping of D into X. For later use we recall that if

$$\| Ax - Ay \| \leqslant q \| x - y \| \quad \text{for all } x \text{ and } y \text{ in } D \text{ and } 0 < q \leqslant 1,$$

then A is *strictly contractive on* D if $q < 1$ and *contractive (nonexpansive) on* D if $q = 1$; A is *completely continuous* if A is continuous on D and $A(D)$ is precompact; A is *strongly continuous* if $\{x_n\} \subset D$, $x \in D$ and $x_n \rightharpoonup x$ imply $Ax_n \to Ax$; A is *strongly closed* if $\{x_n\} \subset D$, $x_n \rightharpoonup x$ and $Ax_n \to g$ for some $g \in X$ imply that $x \in D$ and $Ax = g$.

It was shown by Krasnoselsky [10] that if $A = S + T$ with S strictly contractive on D and T completely continuous on D, then A has a fixed point in D provided

$$Sx + Ty \in D \quad \text{for all} \quad x \quad \text{and} \quad y \text{ in } D. \tag{K}$$

Krasnoselsky's result was extended in various directions in the papers of Browder [2] and Petryshyn and Tucker [20] (see also Nashed and Wong [11] for another extension).

By using the theory of projectionally-compact (P-compact) operators developed by the present author in [13, 14, 15] for Banach spaces X having projectionally complete systems, among other results it was shown in [20; Theorem 5.1] that if S is strictly contractive on all of X and T is completely continuous on X, then $A = S + T$ has a fixed point in D provided that on the boundary ∂D of D the operator A satisfies the rather weak condition

$$\textit{If for some } x \textit{ in } \partial D \textit{ equation } Ax = \alpha x \textit{ holds then } \alpha < 1. \tag{$\Pi_1^<$}$$

By assuming additionally that X is uniformly convex and that X has a weakly continuous duality mapping[1] J, it was also shown in [20, Theorem 5.7] that if S is contractive on X, T strongly continuous on X and $A = S + T$ satisfies condition $(\Pi_1^{<})$ on ∂D, then A has a fixed point in D.

In [2] Browder showed that Krasnoselsky's theorem no longer holds even for separable Hilbert spaces when we assume that S is only contractive on D or even on all of X and T is completely continuous; however, when X is a reflexive Banach space with a weakly continuous duality mapping J, S contractive on all of X, T strongly continuous on X and

$$Sx + Ty \in D \quad \text{for all } x \text{ and } y \text{ in } X, \tag{B}$$

then $A = S + T$ has a fixed point in D. The preceding result is a special case of a more general fixed point theorem proven by Browder [2] for weakly semicontractive (and semicontractive) mappings A defined on X under suitable conditions on X and A, where A is *semicontractive (weakly semicontractive)* if there is a map $B = B(x, y)$ of $X \times X$ into D such that $Ax = B(x, x)$ for x in D while for fixed x in X, $B(\cdot, x)$ is contractive on X and $B(x, \cdot)$ is strongly continuous (completely continuous). The proofs of [2] employ the techniques and theory of J-monotone operators.

In [5] Browder and DeFigueiredo, using also the techniques of J-monotone operators, proved a variant of a general fixed-point theorem for the operator $T = I - A$, where A is a J-monotone mapping of X into X which satisfies certain conditions on the boundary of a ball.

The restriction of the above-mentioned results obtained in [2, 5, 20] is that, unlike the conditions and proofs in [10], their proofs require that A or S and T be defined and possess the corresponding properties on *all* of X while the conclusion (i.e., the existence of fixed points) is made only for the set D. In case of a Hilbert space H this restriction may be lifted by extending the corresponding mappings to all of H but in arbitrary Banach spaces such extension theorems are not available and the question of how to eliminate this restriction remains open.

The purpose of these remarks is threefold. First, in Section 1, we prove two general fixed-point theorems for the mappings $A = S + T$ satisfying the condition $(\Pi_1^{<})$ on ∂D under the assumption that S and T are defined *only* on D and are such that S is strictly contractive on D and T is completely continuous on D or S is contractive on D and T is strongly continuous. From these theorems we then deduce the sharpened versions of theorems in Petryshyn-Tucker [20] quoted above for mappings defined only on D. We

[1] For the definitions of various concepts and the precise statements of the results mentioned in the Introduction in all their generality, see the succeeding sections or the original papers [2, 10, 14, 20].

add that for the class of spaces considered in this section, as a byproduct, we also deduce from our Theorem 2 below a new and thus far the most general fixed-point theorem for contractive mappings defined on $D \subset X$. Second, in Section 2, we show that, under slightly more stronger condition on X, the fixed-point theorem in Browder [2; Theorem 2] remains valid for $B(x, y)$ defined only on $D \times D$. In our discussion in this section we follow the arguments of Browder [2] and those of the author [16]. Third, in Section 3, under the strengthened assumption on X, we show that the main theorem in Browder-De Figueiredo [5] remains valid when A is assumed to be defined only on D. Our proof is somewhat different from the one given in [5].

Finally, we remark that the arguments of Sections 1 and 3 are very much facilitated by the introduction of the so-called *Condition* (c) which was introduced by the author in [17] for the study of the projection method in the solution of nonlinear equations. (For further remarks concerning this condition see [3, 18]). This condition allows us not only to obtain sharpened versions of a number of known results but also to derive some new ones.

Section 1

Let X be a real Banach space and let "$\rightarrow$" and "$\rightharpoonup$" denote the *strong* and the *weak* convergence in X, respectively. Following Fan and Glickberg [7] we say that X has Property (H) if X is strictly convex and if the relations $x_n \rightharpoonup x$ and $\| x_n \| \rightarrow \| x \|$ imply $x_n \rightarrow x$. This and many other equivalent properties in X have been studied in [7], where it was also shown that Hilbert spaces, uniformly convex and locally uniformly convex Banach spaces are examples of spaces having Property (H); however, a Banach space having Property (H) is not necessarily reflexive. Unless stated otherwise, we assume in this paper that X is a real Banach space and that D is a closed ball about the origin X of radius $r > 0$. For later use we need the following simple but useful lemma.

Lemma 1. *If X has Property (H) and $\{x_n\} \subset D$ is a sequence such that $x_n \rightharpoonup x$, then $x \in D$ and either $x \in \operatorname{Int}(D)$ or $x_n \rightarrow x$.*

Proof. Suppose $\{x_n\} \subset D$ and $x_n \rightharpoonup x$. Then $x \in D$, since D is weakly closed, and either $x \in \operatorname{Int}(D)$ or $x \in \partial D$. If $x \in \partial D$, then

$$r = \| x \| \leqslant \liminf_n \| x_n \|$$

since $x_n \rightharpoonup x$. On the other hand, because $\| x_n \| \leqslant r$, $\lim_n \sup \| x_n \| \leqslant r$. Together, these two inequalities imply that $\lim_n \| x_n \| = r = \| x \|$. Since X has Property (H), $x_n \rightarrow x$.

In the rest of this section we assume that X is a separable Banach space with a projectionally complete system $(\{X_n\}, \{P_n\})$, where each X_n is a finite dimensional subspace of X and P_n is a linear projection of X onto X_n such that $P_n x \to x$ for each x in X. We assume additionally that $\| P_n \| = 1$ for each n. We note that separable Hilbert spaces, Banach spaces with monotone Schauder basis and the spaces $C[a, b]$ (of continuous functions with sub-norm) and $L^p(m)$ ($1 < p < \infty$ and for a σ-finite measure m) all have the above properties. In fact, the above spaces possess the somewhat more restrictive Property (Π_1)[2] (see [5, 6, 19]).

The class of operators to be employed in this section is given by

DEFINITION [19]. *A mapping A of D into X is called a generalized P-compact operator or P_γ-compact if $P_n A$ is continuous in $X_n \cap D$ for all sufficiently large n and if there exists a constant $\gamma \geqslant 0$ such that for any p dominating γ (i.e., $p \geqslant \gamma$ if $\gamma > 0$ and $p > \gamma$ if $\gamma = 0$) and any sequence $\{x_n \mid x_n \in X_n \cap D\}$ the strong convergence of the sequence $\{P_n A x_n - p x_n\}$ implies the existence of a subsequence $\{x_{n_i}\}$ and an element x in D such that $x_{n_i} \to x$ and $P_{n_i} A x_{n_i} \to Ax$.*

REMARK 1. It was pointed out in [19] (for details and justification see [20]) that the concept of P_γ-compactness is a generalization of the concept of P-compactness introduced and studied in [13-15] and that the fixed-point theorems obtained there are valid (no change in the proof is necessary) for P_1-compact mappings of D into X.

Our first result is the following theorem:

THEOREM 1. *Suppose X is reflexive and $A = S + T$ maps D into X, where S is strictly contractive on D and T is completely continuous on D. Suppose also that for each fixed $\gamma > 0$ and any $k \geqslant \gamma$ the mapping $S_k = S - kI$ satisfies the following condition (c):*

For any subsystem $(\{X_m\}, \{P_m\})$ and any sequence $\{x_m \mid x_m \in X_m \cap D\}$, the relation $x_m \rightharpoonup x$ and $P_m S_k x_m \to h$ implies that $S_k x = h$.

If for any fixed $\mu \geqslant \gamma$ the operator $A = S + T$ satisfies the condition:

$(\Pi_\mu^{<})$ If the equation $Ax = \alpha x$ holds for some $x \in \partial D$ then $\alpha < \mu$, then there exists $u \in (D - \partial D)$ such that $Su + Tu = \mu u$.

PROOF. In virtue of Theorem 2 in [13] (see also [20]) and Remark 1, to prove Theorem 1 it suffices to establish the following lemma:

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[2] A separable Banach space X is said to have the *Property* (Π_1) if the system $(\{X_n\}, \{P_n\})$, where X_n is a finite dimensional subspace of X and P_n is a linear projection of X onto X_n, is such that $X_n \subset X_{n+1}$ $(n = 1, 2,\ldots)$, $\overline{\bigcup_n X_n} = X$, $\| P_n \| = 1$ $(n = 1, 2,\ldots)$ and $P_n P_j = P_j$ for $n \geqslant j$.

LEMMA 2. *If X and $A = S + T$ are as in Theorem 1 and S is such that $S_k = S - kI$ satisfies condition* (c), *then A is P_γ-compact for any fixed $\gamma > q$.*

PROOF OF LEMMA 2. Since T is completely continuous, the result in [14] implies that to prove Lemma 2 it suffices to show that S is P_γ-compact.

Let $\{x_n \mid x_n \in X_n \cap D\}$ be any sequence such that for some $p \geqslant \gamma$

$$g_n \equiv P_n S x_n - p x_n \to g \qquad \text{for some} \qquad g \text{ in } X.$$

Because X is reflexive, $\{x_n\}$ is bounded and D is weakly closed there exists a subsequence $\{x_m\}$ of $\{x_n\}$ and an element x in D such that $x_m \rightharpoonup x$, and of course, $g_m = P_m S x_m - p x_m \to g$. Hence condition (c) with $k = p \geqslant \gamma$ implies that $Sx - px = g$. Put $L = S - pI$ and observe that since $\| P_m \| = 1$, $p \geqslant \gamma > q$ and S is strictly contractive on D with ratio $q < 1$, $P_m x \in D$ and for $m \geqslant 1$

$$\begin{aligned} \| P_m L P_m x - P_m L x_m \| &\geqslant p \| P_m x - x_m \| - \| P_m (S P_m x - S x_m) \| \\ &\geqslant (\gamma - q) \| P_m x - x_m \| \end{aligned} \tag{1.1}$$

with $\gamma - q > 0$. Since $P_m L x_m \to g$, $P_m L P_m x \to Lx$ and $Lx = g$, the passage to the limit in (1.1) implies that $\| P_m x - x_m \| \to 0$ i.e., $x_m \to x$. This and the continuity of S show that $P_m S x_m \to Sx$. Hence S and, consequently, $A = S + T$ is P_γ-compact.

REMARK 2. If we assume additionally that X has Property (H), then instead of the condition $\| P_n \| = 1$ it is sufficient to assume that $\| P_n \| < (\gamma/q)$ for all n. Since, as was observed in [7], every reflexive Banach space with a basis can be renormed so as to satisfy the Property (H), it follows that for spaces X we consider there is not real loss of generality whenever we assume that X has Property (H).

COROLLARY 1. *If $A = S + T$ satisfies the conditions of Theorem 1 for $\mu = \gamma = 1$ $(> q)$, then A has a fixed point in $(D - \partial D)$.*

REMARK 3. Let us observe that the crucial condition (c) imposed on S_k is precisely the condition (c) used by the author [17] in the study of projectional solvability of nonlinear functional equations.

If in Theorem 1 we drop the hypothesis that X is reflexive, we must modify condition (c) as follows.

THEOREM 2. *Suppose that instead of the reflexivity of X and of the condition* (c) *of Theorem 1, we assume that S_k satisfies the following condition* (h):

For any subsystem $(\{X_m\}, \{P_m\})$ *and any sequence* $\{x_m \mid x_m \in X_m \cap D\}$ *the relation* $P_m S_k x_m \to h$ *implies the existence of an element* x *in* D *such that* $S_k x = h$.

Then the assertions of Theorem 1 and Corollary 1 remain valid.

PROOF. As before, Theorem 2 follows from Theorem 2 in [13] and the sufficiency part of the following lemma.

LEMMA 3. *When* X, S *and* T *are as in Theorem 2, then* $A = S + T$ *is* P_γ*-compact on* D *for any fixed* $\gamma > q$ *if and only if condition* (h) *holds.*

PROOF OF LEMMA 3. Suppose that condition (h) holds and assume that $\{x_m \mid x_m \in X_n \cap D\}$ is any sequence such that for any $p \geqslant \gamma$

$$g_m \equiv P_m S x_m - p x_m \to g \qquad \text{for some} \qquad g \text{ in } X.$$

By condition (h) with $k = p$, there exists an element x in D such that $Sx - px = g$. As before, putting $L = S - pI$, we find that

$$(\gamma - q) \| x_m - P_m x \| \leqslant \| P_m L x_m - P_m L P_m x \| \to 0 \qquad (m \to \infty).$$

Consequently, $x_m \to x$ and $P_m S x_m \to Sx$, i.e., S and, hence, $A = S + T$ is P_γ-compact.

CONVERSE. Suppose that $A = S + T$ is P_γ-compact and that for any given $k \geqslant \gamma$ and $S_k = S - kI$

$$g_n \equiv P_n S_k x_n \to g \qquad \text{for some} \qquad g \text{ in } X. \tag{1.2}$$

Since T is completely continuous and A is P_γ-compact, then so is $A - T$, i.e., S is also P_γ-compact. Hence there exists a subsequence $\{x_m\}$ of $\{x_n\}$ and an element x in D such that $x_m \to x$ and $P_m S x_m \to Sx$. This and (1.2) imply that $Sx - kx = g$, i.e., condition (h) is satisfied.

THEOREM 3. *Suppose* X *is reflexive and* $A = S + T$ *maps* D *into* X, *where* S *is contractive on* D *and* T *is strongly continuous on* D. *Suppose also that* $S_k = S - kI$ *satisfies condition* (c) *for each* $k \geqslant 1$. *If* $A = S + T$ *satisfies condition* $(\Pi_1^{<})$ *on* ∂D, *then* A *has a fixed point in* $(D - \partial D)$.

PROOF. To prove Theorem 3 we first establish the following lemma.

LEMMA 4. *If* X *and* A *are as in Theorem 3 and* S_k *satisfies condition* (c) *for each fixed* $k \geqslant 1$, *then* $A - I$ *is* P_0*-compact on* D.

PROOF OF LEMMA 4. Note first that since X is reflexive and T is strongly continuous on D, to prove Lemma 4, it suffices to show that $S - I$ is P_0-compact.

Let $\{x_n \mid x_n \in X_n \cap D\}$ be any sequence such that for $p > 0$

$$g_n \equiv P_n(S - I)\, x_n - p x_n \to g \qquad \text{for some} \qquad g \text{ in } X. \tag{1.3}$$

Since X is reflexive and D is weakly closed, without loss of generality, we may assume that $x_n \rightharpoonup x$ and $x \in D$. It follows from (1.3) that, for $k = 1 + p > 1$, $P_n S x_n - k x_n \to g$. Hence, by condition (c), $Sx - kx = g$. Put $L = S - kI$ and observe that, since $\| P_n \| = 1$ and S is contractive on D, $P_n x \in D$ and for $n \geqslant 1$

$$\begin{aligned} \| P_n L P_n x - P_n L x_n \| &\geqslant (1 + p) \| P_n x - x_n \| - \| P_n(S P_n x - S x_n) \| \\ &\geqslant p \| P_n x - x_n \| . \end{aligned}$$

Since $P_n L P_n x \to Lx$, $P_n L x_n \to g$ and $Lx = g$, we deduce from the above inequality that $x_n \to x$. This and the continuity of $(S - I)$ imply that $(S - I)$ and, therefore, $(S + T - I)$ is P_0-compact.

To complete the proof of Theorem 3 note first that if $(A - I)\, x = \alpha x$ for some x in ∂D then, in virtue of condition $(\Pi_1^{<})$ satisfied by A on ∂D, it follows that $\alpha < 0$. Hence Lemma 4 and Theorem 2 in [13] (see also [20]) imply that for any $\mu_s > 0$ (with $\mu_s \to 0$ as $s \to \infty$) there exists an element y_s in $(D - \partial D)$ such that

$$(A - I)\, y_s = (S + T)\, y_s - y_s = \mu_s y_s . \tag{1.4}$$

Since X is reflexive and $\{y_s\}$ is bounded, without loss of generality, we may assume that $y_s \rightharpoonup y$ as $s \to \infty$ where y is some element of D. Now by strong continuity of T, $Ty_s \to Ty$. This, (1.4) and the fact that $\mu_s \to 0$ as $s \to \infty$ imply that

$$S y_s - y_s = \mu_s y_s - T y_s \to - Ty. \tag{1.5}$$

Since $(\{X_n\}, \{P_n\})$ is projectionally complete in X, for each s and $\epsilon_s = (1/s)$, there exists an integer $n(s)$ (which we can and shall assume that $n(s) > s$) such that

$$\| y_s - w_{n(s)} \| < \epsilon_s , \qquad w_{n(s)} = P_{n(s)} y_s . \tag{1.6}$$

Thus, (1.5) and (1.6) imply that $w_{n(s)} \in X_{n(s)} \cap D$, $w_{n(s)} \rightharpoonup y$ as $s \to \infty$ and

$$P_{n(s)} S w_{n(s)} - w_{n(s)} \to - Ty.$$

Hence, by condition (c) for $k = 1$, it follows that $Sy - y = - Ty$, i.e., y is a fixed point of $A = S + T$ in D. Q.E.D.

It follows from Theorems 1 and 3 that for general reflexive Banach spaces the condition (c) imposed on S or S_k plays an important role in the derivation of the fixed point theorems for the mappings $A = S + T$. Therefore, it is of interest to see what additional conditions on X would imply the fulfillment of condition (c). Lemmas 5 and 6 below treat this problem.

Let X^* denote the conjugate space of X and let (x, f) denote the value of the linear functional f in X^* at the element x of X. Let $\mu(r)$ be a continuous strictly increasing real-valued function on reals R_1 with $\mu(0) = 0$. A mapping J of X into X^* (see [2, 5]) is called a *duality mapping* with a gauge function μ if

$$(Jx, x) = \| Jx \| \| x \| \qquad \text{and} \qquad \| Jx \| = \mu(\| x \|)$$

for each x in X.

It was shown in [2, 5] that if X^* is strictly convex, J is uniquely determined by its gauge function μ and exactly one duality map J exists for each μ; moreover, J is continuous from the strong topology of X to the weak * topology of X^* and $P_n^* Jx = Jx$ for every x in X_n, where P_n^* denotes the adjoint mapping of P_n which is an idempotent selfmapping of X^*. If in addition X^* is uniformly convex, J is continuous. However, the uniform convexity of X and X^* does not imply the weak continuity of J (i.e., $x_n \rightharpoonup x$ in X does not imply $Jx_n \rightharpoonup Jx$ in X^*) for any duality mapping J of X into X^*. But, for example, when $X = H$, a Hilbert space, then X^* can be identified with H by the inner product and the simplest duality mapping is the identity mapping $J = I$; when X is the Banach space ℓ_p $(1 < p < \infty)$ then $X^* = \ell p/(p-1)$ and the simplest duality mapping is the canonical mapping

$$J(\{x_n\}) = \{| x_n |^{p-2} x_n\}.$$

In both these examples, J is both continuous and weakly continuous. Duality mappings constitute a useful tool in the study of various classes (contractive, J-monotone, accretive) of operators (see [2, 3, 5, 12, 17, 19]).

DEFINITION [3]. *If A is a mapping of D into X and J is a duality mapping of X into X^*, then A is said to be accretive (or J-monotone on D) if*

$$(Ax - Ay, J(x - y)) \geqslant 0 \qquad \textit{for all } x \textit{ and } y \textit{ in } D.$$

LEMMA 5. *Suppose X is reflexive, X^* strictly convex, and X has a weakly continuous duality mapping J. If S is a strictly contractive mapping of D into X with ratio $q < 1$ and γ is any fixed number such that $\gamma > q$, then for any $k \geqslant \gamma$ the operator $S_k = S - kI$ satisfies conditon* (c).

PROOF. Let $\{x_m \mid x_m \in X_m \cap D\}$ be any sequence such that $x_m \rightharpoonup x$ and for any given $k \geqslant \gamma$

$$g_n \equiv P_m S x_m - k x_m \to g \qquad \text{for some} \qquad g \text{ in } X.$$

Since $P_m^* Jx = Jx$ for any x in X_m and $S_k = S - kI$, the strict contractivity of S implies that

$$| (P_m S_k P_m x - P_m S_k x_m, J(P_m x - x_m)) |$$
$$= | (S_k P_m x - S_k x_m, J(P_m x - x_m)) |$$
$$\geqslant k \| P_m x - x_m \| \| J(P_m x - x_m) \| - q \| P_m x - x_m \| \| J(P_m x - x_m) \|$$
$$\geqslant (k - q) \| P_m x - x_m \| \mu(\| P_m x - x_m \|)$$
$$\geqslant (\gamma - q) \| P_m x - x_m \| \mu(\| P_m x - x_m \|). \tag{1.7}$$

Now, because $P_m S_k P_m x - P_m S_k x_m \to S_k x - g$, $P_m x - x_m \rightharpoonup 0$ and J is weakly continuous, the passage to the limit in (1.7) as $m \to \infty$ yields

$$| (P_m S_k P_m x - P_m S_k x_m, J(P_m x - x_m)) | \to | (S_k x - g, 0) = 0,$$

i.e.,

$$\| P_m x - x_m \| \mu(\| P_m x - x_m \|) \to 0.$$

Since $\mu(r)\, r$ is a strictly increasing function, the last relation shows that $\| P_m x - x_m \| \to 0$ or that $x_m \to x$ as $m \to \infty$. This and the continuity of S imply that $S_k x_m \to S_k x$ and, therefore, $g_m = P_m S_k x_m \to S_k x = g$, i.e., condition (c) holds.

In virtue of Lemma 5, as a special case of Theorem 1, we deduce the following new result.

THEOREM 4. *Suppose X is reflexive, X^* strictly convex and X has a weakly continuous duality mapping J. If $A = S + T$ is a mapping of D into X which is such that S is strictly contractive on D, T completely continuous on D and A satisfies the condition $(\Pi_1^{<})$ on ∂D, then $A = S + T$ has a fixed point in $(D - \partial D)$.*

Since, as was noted above, the spaces H and ℓ_p $(1 < p < \infty)$ have weakly continuous duality mappings, Theorem 4 gives the following corollary.

COROLLARY 2. *If X is either H or ℓ_p and $A = S + T$ is a mapping of D into X such that S is strictly contractive on D, T completely continuous and A satisfies condition $(\Pi_1^{<})$ on ∂D, then A has a fixed point in D.*

REMARK 4. For the case of a Hilbert space $X = H$, Corollary 2 was proved in [20] under the same condition $(\Pi_1^{<})$ on ∂D. For the case $X = \ell_p$ $(1 < p < \infty)$, Corollary 2 is new.

LEMMA 6. *Let X be a reflexive Banach space such that X^* is strictly convex and X has Property (H) and a weakly continuous duality mapping J. If S is a contractive mapping of D into X, then for any $k \geqslant 1$ the mapping $S_k = S - kI$ satisfies condition* (c).

PROOF. Let $\{x_m \mid x_m \in X_m \cap D\}$ be any sequence such that $x_m \rightharpoonup x_0$ and for any fixed $k \geqslant 1$ and some g in X

$$g_m = P_m S x_m - k x_m \to g.$$

Since D is weakly closed, $x_0 \in D$ and either $x_0 \in \operatorname{Int}(D)$ or $x_0 \in \partial D$. Suppose first that $x_0 \in \partial D$. Then, by Lemma 1, $x_m \to x_0$ and, consequently, $P_m S x_m \to S x_0$. This implies that

$$g_m = P_m S_k x_m = P_m S x_m - k x_m \to S x_0 - k x_0 = g.$$

Suppose now that $x_0 \in \operatorname{Int}(D)$. Since $k \geqslant 1$ and S is contractive on D, $L = kI - S = -S_k$ is accretive on D. Indeed, for any x and y in D we have

$$\begin{aligned}(Lx - Ly, J(x-y)) &= k(x-y, J(x-y)) - (Sx - Sy, J(x-y)) \\ &\geqslant (k-1)\,\|x-y\|\,\mu(\|x-y\|) \geqslant 0. \qquad (1.8)\end{aligned}$$

Since $P_m^* Jx = Jx$ for any x in X_m and $P_m y \in D \cap X_m$ for any y in D, it follows from (1.8) that

$$(P_m L P_m y - P_m L x_m\,, J(P_m y - x_m)) \geqslant 0. \qquad (1.9)$$

Since $P_m y - x_m \rightharpoonup y - x_0$, J is weakly continuous and $-P_m L x_m \to g$, the passage to the limit in (1.9) gives

$$(Ly + g, J(y - x_0)) \geqslant 0 \qquad \text{for each } y \text{ in } D. \qquad (1.10)$$

The inequality (1.10) implies that $Lx_0 + g = 0$ or $S_k x_0 = g$ because assuming to the contrary that $Lx_0 + g \neq 0$ we arrive (see [16] for $X = H$) at the contradiction. Indeed, if $Lx_0 + g \neq 0$, then since J maps X onto X^* (see [2]), there exists an element z in X such that

$$(Lx_0 + g, Jz) < 0. \qquad (1.11)$$

Since $x_0 \in \operatorname{Int}(D)$, for $t > 0$ and sufficiently small, $x_t = x_0 + tz$ lies in D and the replacement of y in (1.10) by x_t gives

$$(Lx_t + g, J(tz)) \geqslant 0. \qquad (1.12)$$

But, as was noted in [2], $J(tz) = \beta_z(t)\, Jz$ with $\beta_z(t) > 0$ for $t > 0$. Hence, (1.12) implies that

$$(Lx_t + g, Jz) \geqslant 0 \qquad (1.13)$$

Adding $-(Lx_0 + g, Jz)$ to both sides of (1.13) and taking into account (1.11) we get

$$(Lx_t - Lx_0\,, Jz) \geqslant -(Lx_0 + g, Jz) > 0. \qquad (1.14)$$

Since the right-hand side of (1.14) is independent of t and L is continuous, the passage to the limit in (1.14) as $t \to 0^+$ gives the contradiction

$$0 = (Lx_0 - Lx_0, Jz) \geqslant -(Lx_0 + g, Jz) > 0.$$

This proves the validity of Lemma 6.

In virtue of Lemma 6, as a special case of Theorem 3, we deduce the following new result.

THEOREM 5. *Suppose X is reflexive, X^* is strictly convex and X has Property (H) and a weakly continuous duality mapping J. If $A = S + T$ is a mapping of D into X which is such that S is contractive on D, T is strongly continuous on D and A satisfies the condition $(\Pi_1^<)$ on ∂D, then $A = S + T$ has a fixed point in D.*

If in Theorem 5 we take $T = 0$, then we obtain the following interesting new special result.

COROLLARY 3. *Suppose X is reflexive, X^* is strictly convex and X has Property (H) and a weakly continuous duality mapping J. Let S be a contractive (nonexpansive) mapping of D into X which satisfies condition $(\Pi_1^<)$ on ∂D. Then S has a fixed point in $(D - \partial D)$.*

COROLLARY 4. *Let X be either a Hilbert space H or a Banach space ℓ_p $(1 < p < \infty)$ and let $A = S + T$ be a mapping of D into X such that S is contractive on D, T is strongly continuous on D and $A = S + T$ satisfies the condition $(\Pi_1^<)$ on ∂D. Then A has a fixed poiht in $(D - \partial D)$.*

REMARK 5. For the class of our Banach spaces X, Corollary 3 represents a new and thus far the most general fixed-point theorem for a nonexpansive mapping which is defined only on D and which is required to satisfy only a very mild condition $(\Pi_1^<)$ on ∂D; indeed, the fixed point theorem of Browder-Kirk-Gohde [4, 9, 8] for general uniformly convex Banach spaces requires the nonexpansive mapping A to map D into D. We note that since condition $(\Pi_1^<)$ on ∂D is equivalent to Leray-Schauder condition on ∂D, Corollary 4 for $X = H$ and $T = 0$ was first proved by Browder [1] by employing the theory and techniques of monotone operators defined on all of H. For the case when S is contractive on all of X and T is strongly continuous on all of X, Theorem 5 was first established in Petryshyn-Tucker [20].

Added in proof: Corollaries 2 and 4 for the case when X is a Hilbert space H and $A = S + T$ maps $D \subset H$ into D have been also obtained by Zabrejko, Kachurowsky and Krasnoselsky (see also Frum-Ketkow and Sadowsky) in their recent paper published in *Functional Analysis and Its Applications* **1**

(1967), 93–94, which came to our attention after the manuscript of the present paper had been accepted for publication.

SECTION 2

As was already noted in the Introduction, the fixed-point theorems of Browder [2] constitute, in effect, an extension to more general mappings in suitable Banach spaces of Krasnoselsky's theorem under the hypothesis that the mappings are defined on all of X and satisfy a condition analogous to condition (B) above. The purpose of these remarks is to show that if in addition to conditions on X imposed in [2] we assume that X has Property (H), then the results of Browder [2] remain valid without the assumption that the mappings are defined and have the corresponding properties on all of X. We add that Lemma 7 and Proposition 1 below represent an extension and sharpening of the corresponding results in [2, 12] and they possess an interest in their own right. In this section we consider the following class of operators (cf. Browder [2]).

DEFINITION. *A is said to be a semicontractive (weakly semicontractive) mapping of D into X if there exists a mapping B of $D \times D$ into X such that $Ax = B(x, x)$ for x in D while for fixed x in D, $B(\cdot, x)$ is contractive on D and $B(x, \cdot)$ is strongly continuous (completely continuous) on D.*

For later use we prove the following lemma.

LEMMA 7. *Let X be a reflexive Banach space with Property (H) and with a weakly continuous duality mapping J. If A is a semicontractive mapping of D into X, then the mapping $C = I - A$ is strongly closed.*

PROOF. Let $\{x_n\}$ be a sequence in D so that $x_n \rightharpoonup x_0$ and $Cx_n \to w$ for some w in X. First note that $x_0 \in D$ since D is weakly closed. To prove that $Cx_0 = w$ suppose first that $x_0 \in \partial D$. Then, by Lemma 1, $x_n \to x_0$ and, therefore, $B(x_n, x_n) = Ax_n = x_n - Cx_n \to x_0 - w$. Furthermore, the properties of B imply that

$$\begin{aligned} &\| B(x_n, x_n) - B(x_0, x_0) \| \\ &\leqslant \| B(x_n, x_n) - B(x_0, x_n) \| + \| B(x_0, x_n) - B(x_0, x_0) \| \\ &\leqslant \| x_n - x_0 \| + \| B(x_0, x_n) - B(x_0, x_0) \| \to 0 \end{aligned}$$

Hence $x_0 - w = B(x_0, x_0) = Ax_0$, i.e., $Cx_0 = w$.

Suppose now that $x_0 \in \operatorname{Int}(D)$ and define the mapping

$$W(y, x) = y - B(y, x) \qquad \text{for all } x \text{ and } y \text{ in } D.$$

Since, for fixed x in D, $B(\cdot, x)$ is contractive on D, it follows that $W(\cdot, x)$ is accretive on D. Indeed, for all u and v in D and any x in D,

$$\begin{aligned}
&(W(u, x) - W(v, x), J(u - v)) \\
&= (u - v, J(u - v)) - (B(u, x) - B(v, x), J(u - v)) \\
&\geqslant \| u - v \| \| J(u - v) \| - \| B(u, x) - B(v, x) \| \| J(u - v) \| \\
&\geqslant \| u - v \| \| J(u - v) \| - \| u - v \| \| J(u - v) \| = 0
\end{aligned} \tag{2.1}$$

Thus, it follows from (2.1) that for any y in D and each $n \geqslant 1$

$$(W(y, x_n) - W(x_n, x_n), J(y - x_n)) \geqslant 0 \tag{2.2}$$

Since J is weakly continuous, B is strongly continuous in its second variable and $y - x_n \rightharpoonup y - x_0$, it follows that $J(y - x_n) \rightharpoonup J(y - x_0)$ and

$$W(y, x_n) = y - B(y, x_n) \to y - B(y, x_0) = W(y, x_0).$$

This and the relation $W(x_n, x_n) = Cx_n \to w$ imply that the passage to the limit in (2.2) yields

$$(W(y, x_0) - w, J(y - x_0)) \geqslant 0 \qquad \text{for each } y \text{ in } D. \tag{2.3}$$

The inequality (2.3) implies that $W(x_0, x_0) = Cx_0 = w$ because assuming to the contrary that $Cx_0 - w \neq 0$ we arrive at the contradiction. Indeed, if $Cx_0 - w \neq 0$, then there exists an element z in X such that

$$(Cx_0 - w, Jz) < 0. \tag{2.4}$$

Since $x_0 \in \operatorname{Int}(D)$, for $t > 0$ and sufficiently small, $x_t = x_0 + tz$ lies in D and the replacement of y in (2.3) by x_t gives the inequality

$$(W(x_t, x_0) - w, J(tz)) \geqslant 0$$

or the inequality

$$(W(x_t, x_0) - w, Jz) \geqslant 0. \tag{2.5}$$

Adding $-(W(x_0, x_0) - w, Jz)$ to both sides of (2.5) we get

$$(W(x_t, x_0) - W(x_0, x_0), Jz) \geqslant -(W(x_0, x_0) - w, Jz) > 0$$

Since the right-hand side is independent of t and $W(x_t, x_0) \to W(x_0, x_0)$ as $t \to 0$, the passage to the limit as $t \to 0^+$ in the last inequality yields the contradiction

$$0 \geqslant -(W(x_0, x_0) - w, Jz) > 0.$$

This completes the proof of Lemma 7.

COROLLARY 5. *Under the conditions of Lemma 7, the set $C(D)$ is closed in X.*

PROOF. Let $\{x_n\} \subset D$ be a sequence such that $Cx_n \to w$. We want to show that $w \in C(D)$. Since X is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_m\}$ of $\{x_n\}$ and an element x_0 in X such that $x_m \rightharpoonup x_0$ and $Cx_m \to w$. Since, by Lemma 7, C is strongly closed we see that $x_0 \in D$ and $Cx_0 = w \in C(D)$.

THEOREM 6. *Let X be a reflexive Banach space with Property (H) and with a weakly continuous duality mapping J. Let $Ax = B(x, x)$ be a weakly semicontractive mapping of D into D such that $(I - A)(D)$ is closed in X. Then A has a fixed point in D.*

PROOF. Let q be a number so that $0 < q < 1$. Then the mapping $A_q(x) = A(qx)$ maps D into D and since X is reflexive and $(I - A)(D)$ is closed in X, it is not hard to show (see [2]) that A has a fixed point in D if A_q does. Thus to prove Theorem 6, it suffices to show that A_q has a fixed point in D for each q such that $0 < q < 1$.

To show this, replace A by A_q for a given $q < 1$ and call again the resulting operator A. Then $Ax = B(x, x)$ where now for each fixed x in D the mapping $B(\cdot, x) = B_x$ is a strictly contractive map of D into D with ratio $q < 1$ and $B(x, \cdot)$ is a completely continuous mapping of D into D. Hence for each fixed x in D, there exists a unique point y in D such that

$$B_x y = B(y, x) = y. \tag{2.6}$$

Thus, Eq. (2.6) defines a mapping R of D into D given by $y = Rx$ which is clearly such that x is a fixed point of A in D if and only if x is a fixed point of R in D. Consequently, in virtue of Schauder fixed-point theorem, it is sufficient to show that R is completely continuous.

To do this, note first that if Q denotes the mapping of $D \times D$ into X given by $Q(x, y) = x - B(x, y)$, then $Rx = y$ if and only if $Q(x, y) = 0$, $Q(z, \cdot)$ is completely continuous and

$$\begin{aligned} &(Q(x, z) - Q(y, z), J(x - y)) \\ &= (x - y, J(x - y)) - (B(x, z) - B(y, z), J(x - y)) \\ &\geqslant \| x - y \| \, \| J(x - y) \| - q \| x - y \| \, \| J(x - y) \| \\ &= (1 - q) \| x - y \| \mu(\| x - y \|) \end{aligned} \tag{2.7}$$

for each triple x, y, and z in D. Now, as was noted in [2], R is completely continuous provided the following is true:

(a) If $\{x_n\} \subset D$ is such that $x_n \to x$ and $y_n = Rx_n \rightharpoonup y$, then $y = Rx$.

(b) If $\{x_n\} \subset D$ is such that $x_n \rightharpoonup x$ and $y_n = Rx_n \rightharpoonup y$, then $y_n \to y$.

We first prove (a). Suppose $\{x_n\} \subset D$, $x_n \to x$ and $y_n = Rx_n \rightharpoonup y$. Since $\{y_n\} \in D$ and $y_n \rightharpoonup y$, either $y \in \partial D$ or $y \in \mathrm{Int}(D)$. If $y \in \partial D$ then, by Lemma 1,

$y_n \to y$. This and the continuity of $B(x, z)$ in the first variable imply that $Q(y_n, z) = y_n - B(y_n, z) \to y - B(y, z)$. Furthermore, since $x_n \to x$ and B is strictly contractive in the first variable and completely continuous (and thus continuous) in the second variable, it follows that

$$\| B(y_n, x_n) - B(y, x) \| \leqslant \| B(y_n, x_n) - B(y, x_n) \| + \| B(y, x_n) - B(y, x) \| \\ \leqslant q \| y_n - y \| + \| B(y, x_n) - B(y, x) \| \to 0. \quad (2.8)$$

Since $y_n = Rx_n$ if and only if $Q(y_n, x_n) = 0$, (2.8) implies that

$$0 = Q(y_n, x_n) = y_n - B(y_n, x_n) \to y - B(y, x) = Q(y, x) = 0,$$

i.e., $Rx = y$.

Suppose now that $y \in \operatorname{Int}(D)$. Note first that, by (2.7) and the equality $Q(y_n, x_n) = 0$, for each z in D and $n \geqslant 1$

$$0 < (Q(z, x_n) - Q(y_n, x_n), J(z - y_n)) = (Q(z, x_n), J(z - y_n)). \quad (2.9)$$

Since $x_n \to x$ and $z - y_n \rightharpoonup z - y$, the continuity of $Q(z, \cdot)$ and the weak continuity of J imply that, on passage to the limit in (2.9) we get

$$(Q(z, x), J(z - y) \geqslant 0 \qquad \text{for each } z \text{ in } D. \quad (2.10)$$

The inequality (2.10) implies that $Q(y, x) = 0$ (i.e., $Rx = y$) because if we assumed that $Q(y, x) \neq 0$ then, since $y \in \operatorname{Int}(D)$, as in the proof of Lemma 7 we would arrive at a contradiction.

To prove (b), suppose $\{x_n\} \subset D$, $x_n \rightharpoonup x$ and $y_n = Rx_n \rightharpoonup y$. It follows from (2.7) that for all n

$$(Q(y_n, x_n) - Q(y, x_n), J(y_n - y)) \geqslant (1 - q) \| y_n - y \| \mu(\| y_n - y \|). \quad (2.11)$$

Since $Q(y_n, x_n) = 0$ and $Q(y, x_n) = y - B(y, x_n)$ lies in a compact subset of X, by choosing a subsequence we may assume that $Q(y, x_n) \to g$ for some g in X. Since $y_n - y \rightharpoonup 0$ and J is weakly continuous, it follows that

$$(Q(y_n, x_n) - Q(y, x_n), J(y_n - y)) \to (-w, 0).$$

This and (2.11) imply that $\| y_n - y \| \mu(\| y_n - y \|) \to 0$ and, because the function $\mu(r) r$ is strictly increasing, $\| y_n - y \| \to 0$, i.e., $y_n \to y$ and, thus, (b) holds. Q.E.D.

Combining Theorem 6 with Corollary 5 we obtain the following fixed-point theorem (cf. Theorem 1 in [2]).

THEOREM 7. *Let X be a reflexive Banach space with Property (H) and with a weakly continuous duality mapping J. If A is a semicontractive mapping of D into D, then A has a fixed point in D.*

Another immediate consequence of Theorem 6 and Lemma 7 is the following theorem (cf. Theorem 4 in [2]).

THEOREM 8. *Let X be a reflexive Banach space with Property (H) and with a weakly continuous duality mapping J. Let A be a semicontractive mapping of D into D and let K be a completely continuous mapping of D into D such that for every sequence $\{x_n\} \subset D$ the condition*

$$x_n \rightharpoonup x \quad \text{and} \quad (x_n - x - B(x_n, x_n) + B(x, x_n), J(x_n - x)) \to 0 \tag{d}$$

implies that $Kx_n \to Kx$. Then the mapping $U = A + K$ has a fixed point in D.

PROOF. By Theorem 6, it suffices to show that $C_0(D)$ is closed in X, where $C_0 = I - U$. Suppose that $\{x_n\} \subset D$ and $C_0 x_n \to w_0$ for some w_0 in X. Since X is reflexive, we may assume that $x_n \rightharpoonup x_0$ with $x_0 \in D$. Thus, it suffices to show that $C_0 x_0 = w_0$. Now, since $B(x, \cdot)$ is strongly continuous and $x_n \rightharpoonup x_0$, it follows that $B(x_0, x_n) \to B(x_0, x_0)$. The complete continuity of K implies that for some subsequence $\{x_m\}$, $Kx_m \to h$ for some h in X while the weak continuity of J implies that $J(x_m - x_0) \rightharpoonup 0$. This and the assumption that $C_0 x_n \to w_0$ shows that

$$\begin{aligned}&(x_m - x_0 - B(x_m, x_m) + B(x_0, x_m), J(x_m - x_0))\\ &= (C_0 x_m + Kx_m - B(x_0, x_m), J(x_m - x_0)) \to (w_0 + h - B(x_0, x_0), 0) = 0.\end{aligned}$$

Thus, our condition (d) on K implies that $Kx_m \to Kx_0 = h$. Consequently, $x_m \rightharpoonup x_0$ and

$$Cx_m = x_m - B(x_m, x_m) = C_0 x_m + Kx_m \to w_0 + Kx_0 .$$

Since by Lemma 7, C is strongly closed we deduce that $Cx_0 = w_0 + Kx_0$, i.e., $C_0 x_0 = w_0$.

REMARK 6. Condition (d) is fulfilled if, for example, either $K(x) = 0$ or $B(x, y) = 0$ for all x and y in D. The first example is that of Theorem 7 and the second is the Schauder fixed-point theorem for our class of Banach spaces.

REMARK 7. Using the same arguments as in the proof of Lemma 6 one can establish the validity of the following proposition which in a certain sense is an extension of the corresponding results in [2, 12].

PROPOSITION 1. *Let X be a reflexive Banach space with Property (H) such that X^* is strictly convex and X has a weakly continuous duality mapping J. If A is either a continuous or a demicontinuous accretive mapping of D into X, then A is strongly closed.*

Section 3

In this section we give a new proof of a sharpened version of the main theorem in Browder-DeFigueiredo [5; Theorem 3] without the assumption that A is J-monotone on all of X.

We recall that a mapping A of D into X is called *demicontinuous* if $\{x_n\} \subset D$, $x \in D$ and $x_n \to x$ imply $Ax_n \rightharpoonup Ax$. Our result in this section is based on the following Lemma.

Lemma 8. *Let X be a reflexive Banach space with Properties (Π_1) and (H) and such that X^* is strictly convex and there exists a duality mapping J of X into X^* which is both continuous and weakly continuous. If A is a demicontinuous accretive mapping of D into X, then A satisfies the condition* (c).

Proof. Let $\{x_n \mid x_n \in X_n \cap D\}$ be any sequence such that

$$x_n \rightharpoonup x_0 \qquad \text{and} \qquad g_n = P_nAx_n \to g \qquad \text{for some } g \text{ in } X.$$

We want to show that $Ax_0 = g$. Note first that, since D is weakly closed, $x_0 \in D$ and either $x_0 \in \partial D$ and, by Lemma 1, $x_n \to x_0$ or $x_0 \in \mathrm{Int}$. In the first case it follows that $Ax_n \rightharpoonup Ax_0$ by demicontinuity of A; furthermore, since Property (Π_1) implies that $P_n^* f \to f$ for every f in X^*, it follows that for each $f \in X^*$

$$(P_nAx_n, f) = (Ax_n, P_n^*f) \to (Ax_0, f),$$

i.e., $P_nAx_n \rightharpoonup Ax_0$. Since, by hypothesis, $P_nAx_n \to g$ it follows that $Ax_0 = g$.

Suppose now that $x_0 \in \mathrm{Int}(D)$. Since $P_n^* Jx = Jx$ for each x in X_n, $X_n \subset X_{n+1}$ for each n, and $P_n{}^2 = P_n$, the accretivity of A on D implies that for any fixed j $(\leqslant n)$ and any z in $X_j \cap D$, $z - x_n \in X_n$ and

$$\begin{aligned} 0 \leqslant (Az - Ax_n, J(z - x_n)) &= (P_nAz - P_nAx_n, J(z - x_n)) \\ &= (Az - g_n, J(z - x_n)). \end{aligned} \tag{3.1}$$

Since $Az - g_n \to Az - g$, $z - x_n \rightharpoonup z - x_0$ and J is weakly continuous, the passage to the limit in (3.1) as $n \to \infty$ implies that for any j and any z in $X_j \cap D$

$$(Az - g, J(z - x_0)) \geqslant 0. \tag{3.2}$$

Since $\| P_n \| = 1$ for each n, $P_jy \in D \cap X_j$ for any y in D and, therefore, (3.2) implies that

$$(AP_jy - g, J(P_jy - x_0)) \geqslant 0. \tag{3.3}$$

By demicontinuity of A, the continuity of J and the fact that $P_j y \to y$ as $j \to \infty$, the passage to the limit in (3.3) shows that

$$(Ay - g, J(y - x_0)) \geqslant 0 \tag{3.4}$$

for each y in D. Since $x_0 \in \mathrm{Int}(D)$, exactly as in the proof of Lemma 6, we deduce from (3.4) that $Ax_0 = g$, i.e., A satisfies condition (c).

REMARK 8. If A is assumed to be continuous instead of demicontinuous, then the assertion of Lemma 8 remains valid without the assumption that J is also continuous.

THEOREM 9. *Let X be a reflexive Banach space with Properties (Π_1) and (H) and such that X^* is strictly convex and there exists a duality mapping J of X into X^* which is both continuous and weakly continuous. Let A be an accretive demicontinuous mapping of D into X such that*

$$(Ax, Jx) \geqslant 0 \qquad \text{for all } x \text{ in } \partial D. \tag{3.5}$$

Then there exists a point x_0 in D such that $Ax_0 = 0$.

PROOF. Consider the mapping $C = I - A$ of D into X. Since C is demicontinuous, for each n the mapping $P_n C : D \cap X_n \to X_n$ is continuous; furthermore, (3.5) implies that $P_n C$ satisfies condition $(\Pi_1^{\leqslant})$ on $\partial D \cap X_n$ for each n. Indeed, suppose that $PC_n x = \alpha x$ for some x in $\partial D \cap X_n$. Then, in view of (3.5) and the fact that $P_n^* Ju = Ju$ for each u in X_n, we get

$$\alpha(x, Jx) = (\alpha x, Jx) = (P_n Cx, Jx) = (Cx, Jx) = (x, Jx) - (Ax, Jx) \leqslant (x, Jx)$$

from which it follows that $\alpha \leqslant 1$. Hence, by Theorem 1 in [14], for each n there exists a vector $x_n \in D \cap X_n$ such that $P_n C x_n = x_n$, i.e., $P_n A x_n = 0$. Since X is reflexive and $\{x_n\}$ is bounded, we may assume by passing to a subsequence that $x_n \rightharpoonup x_0$ for some x_0 in D. Thus, $x_n \rightharpoonup x_0 \in D$ and $g_n = P_n A x_n = 0 \to 0$ as $n \to \infty$. Since, by Lemma 8, A satisfies condition (c) it follows that $Ax_0 = 0$.

REMARK 9. When A in Theorem 9 is assumed to be continuous instead of demicontinuous, then the assertion of Theorem 9 remains valid without the assumption that J is also continuous.

REMARK 10. Theorem 3 in [5] or Theorem 9 above can also be formulated as a fixed-point theorem. In such a reformulation, Theorem 3 in [5] was deduced in [6] as a corollary (Corollary 9 in [6]) of Theorem 1 in [6]. It must be added, however, that since the argument in [6] is based on Proposition 4 in [6] it must be assumed there that the operators in Corollary 9 (as well as in Corollary 10) are defined and have the corresponding properties on all of X.

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